UIET

UTKAL INSTITUTE OF
ENGG & TECHNOLOGY

LECTURER NOTES

ON

(TH.1)

ENGINEERING MATHEMATICS - III

Diploma in Electrical Engineering.

(3rdSemester)

Notes Compiled by: Jayashree Choudhury

Designation:LECTURER IN ELECTRICAL ENGG

Complex Numbers

Real numbers

Integers + Rational numbers + Irrational numbers

E.g. −1, 0, 1, 2, ..., $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, ..., $\sqrt{2}$, $\sqrt{3}$, π, e ... all are combined to form Real number set.

Set of real numbers is denoted by R

Imaginary number

$\sqrt{-1}$ does not belongs to R..

This number is denoted by i, called as the basic imaginary number

Defn

$i = \sqrt{-1}$ = Imaginary number

Formula →

$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

and $i^{4n} = 1$

i.e. $i^4 = i^8 = i^{12} = \cdots = 1$

Using above four formulas we can find other higher powers of i as follows.

E.g.-1

$$i^{10} = i^{(4\times2)+2}$$
$$= i^{(4\times2)} \cdot i^2$$
$$= 1(-1)$$
$$= (-1)$$

Technique

$4 \mid 10 \mid 2$, 8, remainder (2)

Technique → $i^a = i^K$ (where K is the remainder when a is divided by 4)

$4 \mid a \mid b$, 4b, remainder (K)

E.g-2

$i^{31} = i^{3} = -i$

$4 \overline{)31}\,7$, 28, remainder 3

$i^{52} = i^{4\times 13} = 1$

$i^{61} = i^{1} = i$

$4 \overline{)61}\,15$, 4, 21, 20, remainder 1

$\sqrt{-25} = \sqrt{(-1)25} = \sqrt{-1}\sqrt{25}$
$= 5i$

Definition of complex number

The numbers in the form $a+ib$ are called complex numbers where a and b are real numbers.
We denote complex number by notation 'z'.

$Z = a + ib$

a = Real part of z = $Re(z)$

b = Imaginary part of z = $Im(z)$

Set of complex numbers is denoted by C

E.g

① $3+2i$ is a complex number as 3 and 2 are real number
Here Real part = 3
Imaginary part = 2

② $7-\sqrt{2}i$ is a complex number having Real part = 7
Imaginary part = $-\sqrt{2}$

③ $3i$ is also a complex number
As Real part = 0
Imaginary part = 3

④ Every Real number is a complex number. For example
7 is a complex number
its real part = 7
Imaginary part = 0

Note

1) If $Re(z)=0$, Then z is called as purely imaginary number
E.g $3i$, $2i$, $-7i$, πi etc

2) If $Im(z)=0$, Then z is called as a purely real number.
E.g. -1, 3, $\frac{3}{2}$, e, π etc

Conjugate of a complex number

If $z = a+ib$ is a complex number then conjugate of z denoted by $\bar{z}$ is defined as

$\bar{z} = a - ib$

(Just change the sign of imaginary part)

E.g Conjugate of $2+3i$ is
$\overline{2+3i} = 2-3i$

E.g-2

Conjugate of $-2-3i$ is

$\overline{-2-3i} = -2+3i$

E.g-3

Conjugate of $7i$ is

$\overline{7i} = -7i$

E.g-4

Conjugate of -5 is

$\overline{-5} = -5$

Modulus of a complex number

Let $z = a+ib$

then $\boxed{\text{Modulus of } z = |z| = \sqrt{a^2+b^2}}$

E.g

$$|3+2i| = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$$

$$|8-6i| = \sqrt{8^2+(-6)^2} = \sqrt{64+36} = \sqrt{100} = 10$$

Geometrical Representation of complex number

Complex numbers are represented by points on the two dimensional plane.

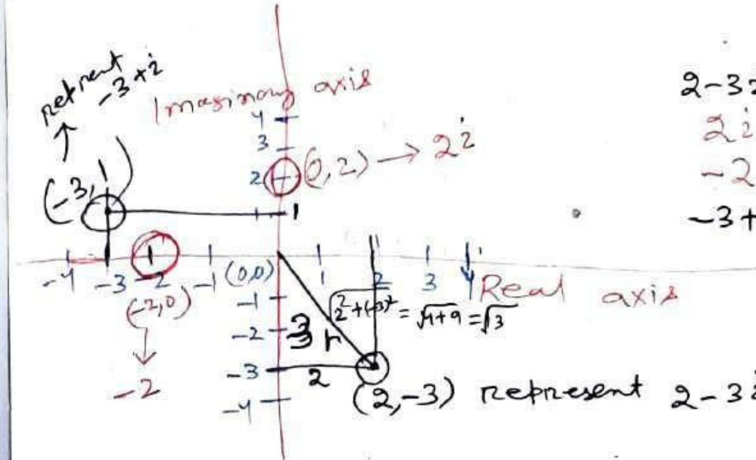
X axis → Real part axis

Y axis → Imaginary Part axis

So $a+ib$ is equivalent to point (a,b) in co-ordinate plane.

$a+ib \longrightarrow (a,b)$

Let us represent $2-3i$, $2i$, -2, $-3+i$ in the plane



$2-3i \to (2,-3)$

$2i \to (0,2)$

$-2 \to (-2,0)$

$-3+i \to (-3,1)$

Polar Form

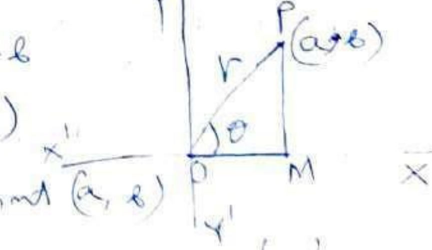
Now let us represent $z = a+ib$ in the Polar form.

Polar form is written as (r, θ).

r = length of line joining origin with the point

θ = angle made by the line with X axis.

We know $z = a+ib$ is represented by (a,b)



Let P be the point (a,b)

Then joing P with origin 'O'.

$\boxed{OP = r}$

and $\boxed{\angle XOP = \theta}$

Then it is clear from figure in OMP $\triangle$, $\cos\theta = \frac{OM}{OP} = \frac{a}{r}$

$\Rightarrow \boxed{a = r\cos\theta}$

Similarly $\sin\theta = \frac{PM}{OP} = \frac{b}{r}$

$\Rightarrow \boxed{b = r\sin\theta}$.

Hence polar representation of $z = a + ib$ is

$z = r\cos\theta + i r\sin\theta$

$\boxed{z = r(\cos\theta + i\sin\theta)}$ → Polar form

Here $\boxed{r = \sqrt{a^2+b^2} = |z| = \text{Modulus of } z}$

θ is called as the amplitude or argument of z written as amp(z) or arg(z).

Def^n of amplitude $\boxed{\theta = \tan^{-1}\frac{b}{a}}$

The angle made by the line joining a complex number with origin with the +ve direction of X axis is called as the amplitude of the complex number (z) & it is denoted by θ.

(5)

Important Note

1) The unique value of θ lies between $-\pi$ to π is called principal value of amplitude.

2) General value of amplitude $= 2n\pi + \theta$ where $n \in \mathbb{Z}$ and θ is the principal value.

E.g.

Q. Find the modulus and amplitudes of a) $z = 1 + i$

b) $z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

a) $z = 1 + i$

Modulus $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Principal $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

General value $\boxed{\theta = 2n\pi + \frac{\pi}{4}}$

Note: When find the principal value we have to follow following rule: 1st find the location of the point (i.e. point present in which Quadrant)

If 1st Quadrant then θ lies $\left(0, \frac{\pi}{2}\right)$

2nd Quadrant → θ lies in $\left(\frac{\pi}{2}, \pi\right)$

3rd Quadrant → θ lies in $\left(-\pi, -\frac{\pi}{2}\right)$

4th Quadrant → θ lies in $\left(-\frac{\pi}{2}, 0\right)$

Clarification (8)

In problem (a), the $z = 1 + i$ lies in 1st Quadrant as $(1, 1)$.

Now $\tan^{-1}(1) = \frac{\pi}{4}$ or $-\frac{3\pi}{4}$

($\frac{\pi}{4}$ ✓ correct; $-\frac{3\pi}{4}$ cancelled as it is applicable for 3rd Quadrant)

So $\theta = \tan^{-1}(1) = \frac{\pi}{4}$

(b) $z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$\text{Modulus} = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

As $z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ represented by point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is in 3rd Quadrant

Hence $\theta \in \left(-\pi, -\frac{\pi}{2}\right)$

Now $\theta = \tan^{-1}\left(\frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)}\right) = \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$ or $-\frac{2\pi}{3}$ ✓

($\frac{\pi}{3}$: ✗ (1st Quadrant); $-\frac{2\pi}{3}$: 3rd Quadrant)

Principal value of $\theta = -\frac{2\pi}{3}$

General value of amplitude $= 2n\pi - \frac{2\pi}{3}$

Question (9)

Represent $1 - \sqrt{3} i$ in its polar form.

Ans For polar form we have to calculate r and θ.

$$r = |z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1^2 + 3} = \sqrt{4} = 2$$

As $1 - \sqrt{3} i$ lies i.e. $(1, -\sqrt{3})$ lies in ~~2nd~~ 4th Quadrant, so θ lies in ~~$(\frac{\pi}{2}, \pi)$~~ $\left(-\frac{\pi}{2}, 0\right)$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \tan^{-1}(-\sqrt{3}) = \cancel{\frac{2\pi}{3}} \text{ or } \left(-\frac{\pi}{3}\right)$$

$= -\frac{\pi}{3}$ (2nd) (4th)

Hence $\theta = -\frac{\pi}{3}$

$$1 - \sqrt{3} i = r(\cos\theta + i \sin\theta)$$

$$= 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right) \text{ (Ans)}$$

$$= 2\left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right)$$

Operations of complex numbers

$z_1 = a + ib$, $z_2 = c + id$

Addition

$z_1 = a + ib$, $z_2 = c + id$

Then $z_1 + z_2 = (a + c) + i(b + d)$

(add real part with real and imaginary part with imaginary part)

E.g. → $(3 + 2i) + (5 - i)$

$= (3 + 5) + (2 + (-1)) i$

$= 8 + i$

Substraction

$z_1 - z_2 = (a-c) + (b-d)i$

E.g.

$(5+2i) - (3+i)$

$= (5-3) + (2-1)i = 2+i$

Multiplication

$z_1 z_2 = (a+ib)(c+id) = a(c+id) + ib(c+id)$

$= ac + iad + ibc + i^2 bd$

$= ac + iad + ibc - bd \quad \{as\ i^2 = -1\}$

$= (ac - bd) + i(ad + bc)$

E.g.

$(2+i)(1-i)$ $\quad \{a=2,\ b=1,\ c=1,\ d=-1$

$= \{2\times1 - 1(-1)\} + i(2\times(-1) + (1\times1))$

$= (2+1) + (-1)i = 3 - i$

Division

$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

$= \frac{ac - iad + ibc - i^2 bd}{c^2 - (id)^2}$

$= \frac{ac - iad + ibc + bd}{c^2 + d^2} \quad (i^2 = -1)$ $\quad \{(id)^2 = i^2 d^2 = -1d^2 = -d^2$

$= \frac{ac+bd}{c^2+d^2} + \frac{(bc - ad)i}{c^2+d^2}$

Example

Question

Find $\frac{2-i}{3+i}$

Ans

$\frac{2-i}{3+i} = \frac{(2-i)(3-i)}{(3+i)(3-i)}$

$= \frac{6 - 2i - 3i + i^2}{3^2 - i^2} = \frac{6 - 5i - 1}{9 - (-1)}$

$= \frac{5-5i}{10} = \frac{5}{10} - \frac{5i}{10}$

$= \frac{1}{2} - \frac{1}{2}i$ (Ans)

Properties of complex numbers

1) $a + ib = 0 \iff a = 0$ and $b = 0$
2) $a + ib = c + id \iff a = c$ and $b = d$
3) $(\bar{\bar{z}}) = z$
4) $(z + \bar{z}) = 2\,Re(z)$
5) $(z - \bar{z}) = 2\,Im(z)$
6) $(z\bar{z}) = |z|^2$ (Important)
7) $Re(z) \le |z|$ and $Im(z) \le |z|$ (Im
8) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
9) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
10) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

11) $|z_1 z_2| = |z_1| \cdot |z_2|$

12) $|z_1 + z_2| \leq |z_1| + |z_2|$

13) $|z_1 - z_2| \leq |z_1| + |z_2|$

Algebra of complex numbers

$z_1, z_2 \in \mathbb{C}$

1) $z_1 + z_2 = z_2 + z_1$ (commutative)

2) $z_1 + z_2 \in \mathbb{C}$

3) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (associative)

4) '0' is the additive Identity

i.e. $z + 0 = 0 + z = z$

5) $z \in \mathbb{C}$, there exist $-z \in \mathbb{C}$ such that $z + (-z) = (-z) + z = 0$

$-z$ is called additive inverse of z.

E.g. $2 + i$ has additive inverse as $-2 - i$

6) $z_1 z_2 \in \mathbb{C}$

7) $z_1 \cdot z_2 = z_2 \cdot z_1$ (Commutative)

8) $(z_1 \cdot z_2) \cdot z_3 = z_1 (z_2 \cdot z_3)$ (associative)

9) '1' is the multiplicative identity

i.e. $z \cdot 1 = 1 \cdot z = z$

10) Multiplicative inverse of nonzero z is $\frac{1}{z}$

i.e. if $z \neq 0$ then $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$

E.g. Multiplicative inverse of $2 + i$ is $\frac{1}{2+i}$ i.e. $\frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4-i^2}$

$= \frac{2-i}{4+1} = \frac{2}{5} - \frac{1}{5} i$

multiplicative inverse of $2+i$ is $\frac{2}{5} - \frac{1}{5} i$

11) **Distributive.**

$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

Some questions on discussed topic

Q.1. Find the value of $(-i)^{4n+1}$

Ans $(-i)^{4n+1} = \{(-1)\, i\}^{4n+1} = (-1)^{4n+1}\, i^{4n+1}$

$= (-1)\, i^{4n} \cdot i^1 = (-1) \times 1 \times i = -i$

Q.2 Find $\frac{1}{i^{15}}$

Ans $\frac{1}{i^{15}}$ (Multiply i on both numerator and denominator ∴ Reason → Make denominator real)

$= \frac{i}{i^{15} \cdot i} = \frac{i}{i^{16}} = \frac{i}{1} = i$ (Ans)

Q.3. Find x, y if $(x - 2y) + 3yi = 6i$

Ans $(x - 2y) + 3yi = 6i$

Equating real parts of both sides

$x - 2y = 0$ — (1)

Equating Imaginary parts we have $3y = 6$

$\Rightarrow \boxed{y = 2}$ — (2)

From (1) and (2)

$x - 2\times 2 = 0$

$\Rightarrow \boxed{x=4}$

Hence $x=4$ and $y=2$

Q.4 Express $\frac{2-i}{(3-i)^2}$ in $a+ib$ form.

Ans

$$\frac{2-i}{(3-i)^2} = \frac{2-i}{3^2-2\times3\times i+i^2} = \frac{2-i}{9-6i-1}$$

$$= \frac{2-i}{8-6i} = \frac{(2-i)(8+6i)}{(8-6i)(8+6i)}$$

$$= \frac{2\times8+2\times6i-i\times8-i\times6i}{8^2-(6i)^2}$$

$$= \frac{16+12i-8i-6i^2}{64-(-36)}$$

$$= \frac{16-4i+6}{100} = \frac{22-4i}{100}$$

$$= \frac{11}{50} - \frac{i}{25} \quad \text{(Ans)}$$

Q.5 Find the multiplicative inverse of $2+i$

Multiplicative inverse of $2+i$ is

$$\frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{2^2-i^2} = \frac{2-i}{4-(-1)}$$

$$= \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5}$$

Q.6 Find the conjugate and modulus of $7-i$.

Conjugate of $7-i = \overline{7-i} = 7+i$

Modulus of $7-i = |7-i| = \sqrt{7^2+(-1)^2}$

$$= \sqrt{49+1} = \sqrt{50}$$

Q.7 Find the modulus and amplitude of $3+4i$.

Modulus $3+4i$ is $= |3+4i| = \sqrt{3^2+4^2}$

$= 5$

Amplitude of $3+4i$ is $\boxed{\theta = \tan^{-1}\left(\frac{4}{3}\right)}$

Q.8 Represent $-1+i$ in polar form.

$z = -1+i$

$r = |z| = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}$

Principal value of argument

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1)$$

{ Here $-1+i$ lies in 2nd Quadrant
So θ lies in $\left(-\pi, -\frac{\pi}{2}\right)$

$$= \frac{3\pi}{4} \text{ and } \frac{-3\pi}{4}$$

(✓) (✗)

Hence $\theta = \frac{3\pi}{4}$

$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (Ans)

Q.9 Prove that $|z_1+z_2| \le |z_1|+|z_2|$

Ans To Prove this we have to prove

$$|z_1+z_2|^2 \le \{|z_1|+|z_2|\}^2$$

As both $|z_1+z_2|$ and $\{|z_1|+|z_2|\}$ are +ve quantities so the above inequality implies the question.

So start from,

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} \quad \left\{\begin{array}{l}\text{See Property} \\ \text{As } |z|^2 = z\bar{z}\end{array}\right\}$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)} + |z_2|^2$$

$$\left\{\begin{array}{l}\text{As } \overline{z_1 z_2} = \bar{z}_1\bar{z}_2 \\ \text{Here } \overline{(z_1\bar{z}_2)} = \bar{z}_1\bar{\bar{z}}_2 = \bar{z}_1 z_2 = z_2\bar{z}_1 \ \text{(Commutative)}\end{array}\right\}$$

$$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$$

$$\{\text{as } z+\bar{z} = 2\,\mathrm{Re}\,z\}$$

$$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \quad \{\text{as } \mathrm{Re}\,z \le |z|\}$$

$$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \quad \{\text{as } |\bar{z}| = |z|\}$$

$$= \{|z_1|+|z_2|\}^2$$

Hence $|z_1+z_2|^2 \le \{|z_1|+|z_2|\}^2$

$$\Rightarrow |z_1+z_2| \le |z_1|+|z_2|$$

Q.10 If $z = x+iy$ show that

$$|x+y| \le \sqrt{2}\,|z|$$

Ans

we know that

$$|z|^2 = z\bar{z}$$

$$= (x+iy)\overline{(x+iy)}$$

$$= (x+iy)(x-iy)$$

$$= x^2 - (iy)^2$$

$$= x^2+y^2$$

Now $|x+y|^2 = (x+y)^2 = x^2+y^2+2xy$

$$\le (x^2+y^2)+(x^2+y^2) \quad \left\{\begin{array}{l}\text{As } (x-y)^2 \ge 0 \\ \Rightarrow x^2+y^2-2xy \ge 0 \\ \Rightarrow x^2+y^2 \ge 2xy\end{array}\right.$$

$$= 2(x^2+y^2)$$

$$= 2|z|^2$$

$$\Rightarrow \boxed{|x+y| \le \sqrt{2}\,|z|} \quad \text{(Proved)}$$

Questions for Practice

1) Express following in a+ib form.

a) $(3+4i)^2$ b) $\dfrac{(2-i)^2}{3+i}$ (c) $\dfrac{1}{i^{21}}$

2) Find the value of x and y if

$(4-5y)+(3-2x)i = 0$

3) Find the modulus and conjugate of

$(2-i)(1+i)$

4) Find the modulus and amplitude of $(1+i)(1+2i)(1+3i)$

5) If z_1 and z_2 are complex numbers, then prove that

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

6) Represent $1-i$ in polar form.

7) Find the following
a) i^{15} b) i^{13} c) i^{71}

8) Find the multiplicative inverse of
a) $5-i$ b) $2+i$

9) Find the additive inverse of
a) $-5+2i$ b) $\frac{7}{2}-\frac{3}{2}i$

10) Write the real and imaginary part of
a) $-3i$ b) $2-\frac{3}{2}i$ c) $\frac{1}{1+i}$
d) $-3-2\sqrt{-1}$

11) Find $(\sqrt{-1})^{32}$?

12) Define modulus and amplitude of a complex number.

Part-2

Square Root of a complex number

Let us evaluate $\sqrt{z}$.

$z \in C$ means $z = a+ib$

Procedure

Now let $\sqrt{a+ib} = x+iy$

Then squaring both sides we have

$$a+ib = (x+iy)^2$$

$$\Rightarrow a+ib = x^2+2ixy+(iy)^2$$

$$\Rightarrow a+ib = x^2 + i(2xy) - y^2$$

$$\Rightarrow a+ib = (x^2-y^2) + i(2xy)$$

Equating real parts and imaginary parts we have

$$x^2-y^2 = a \text{ and } 2xy = b \quad \text{——(1)}$$

Then using formula

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\{\text{i.e. } (a+b)^2 = (a-b)^2 + 4ab\}$$

$$= (x^2-y^2)^2 + (2xy)^2$$

$$= a^2+b^2 \quad \text{——(2)}$$

Now solving (1) and (2) we can get value of x^2 and then value of x.

After getting x using $2xy = b$ we get value of y.

Examples

Questions

Find the square roots of following

(a) $3-4i$

(b) $1+4\sqrt{3}i$

(a) Let $x+iy=\sqrt{3-4i}$.

Squaring both sides,

$$x^2+2ixy+(iy)^2=3-4i$$

$$\Rightarrow (x^2-y^2)+i(2xy)=3-4i$$

$$\Rightarrow \boxed{x^2-y^2=3 \quad \text{and} \quad 2xy=-4.} \quad (1)$$

Now $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$

$$=3^2+(-4)^2=9+16=25$$

$$\Rightarrow \boxed{x^2+y^2=\sqrt{25}=5}$$

{as x^2+y^2 is always +ve

$$x^2+y^2=5$$

$$(+)\quad x^2-y^2=3$$

$$2x^2=8$$

$$\Rightarrow x^2=4 \Rightarrow \boxed{x=\pm 2}$$

When $x=2$

$$2xy=-4$$

$$\Rightarrow y=\frac{-4}{2x}=\frac{-4}{2\times 2}=-1$$

$$\Rightarrow x+iy=2-i$$

When $x=-2$

$$2xy=-4 \Rightarrow y=\frac{-4}{2\times(-1)}=1$$

$$\Rightarrow x+iy=-2+i$$

∴ The square roots of $3-4i$ are $2-i$ and $-2+i$.

b) Let $x+iy=1+4\sqrt{3}i$

Square both sides we have

$$(x^2-y^2)+i(2xy)=1+4\sqrt{3}i$$

real and imaginary part

Equating both sides,

$$x^2-y^2=1 \quad (1)$$

$$2xy=4\sqrt{3} \quad (2)$$

Now

$$(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$$

$$=1^2+(4\sqrt{3})^2=1+48=49$$

$$\Rightarrow x^2+y^2=\sqrt{49}=7$$

$$x^2+y^2=7$$

$$x^2-y^2=1$$

$$2x^2=8 \Rightarrow x^2=4 \Rightarrow x=\pm 2$$

When $x=2$

$$y=\frac{4\sqrt{3}}{2x}=\frac{4\sqrt{3}}{4} \quad (\text{from } (2))$$

$y=\sqrt{3}$ Hence $x+iy=2+\sqrt{3}i$

When $x=-2$

$$y=\frac{4\sqrt{3}}{2x}=\frac{4\sqrt{3}}{-4}=-\sqrt{3}$$

Here $x+iy=-2-\sqrt{3}i$

$$\therefore \sqrt{1+4\sqrt{3}i}=2+\sqrt{3}i \text{ or } -2-\sqrt{3}i$$

Cube roots of Unity

Unity means 1.

Now we have to find $\sqrt[3]{1}$.

Let $x = \sqrt[3]{1}$

Taking cube of both sides

$\Rightarrow x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow x = 1$ or $x = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{-1}\sqrt{3}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

Hence $x = 1$ or $\frac{-1+\sqrt{3}i}{2}$ or $\frac{-1-\sqrt{3}i}{2}$

These are the three cube roots of unity.

Notation

The above three cubes roots of unity are denoted by $1, \omega$ and ω^2. where $\omega = \frac{-1+\sqrt{3}i}{2}$

$\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Formula

1. $\omega^3 = 1$. In general $\omega^{3n} = 1$
2. $1 + \omega + \omega^2 = 0$

Now calculate

$\omega^{17} = \omega^{3\times5+2} = \omega^{3\times5}\cdot\omega^2$ $3\,|\,17\,|\,5$, 15, 2

$= 1\cdot\omega^2 = \omega^2$

Technique $\omega^n = \omega^K$ $3\,|\,n\,|\,m$, $3m$, K

where K is the remainder when n is divided by 3.

Question

Find ω^{101} $3\,|\,101\,|\,33$, 9, 11, 9, 2

Ans = $\omega^{101} = \omega^2$

Find ω^{73}

$\omega^{73} = \omega$ (Ans) $3\,|\,73\,|\,24$, 6, 13, 12, 1

Question

1. If ω is the cube root of unity, then Find $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

Ans $(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= \{(1-\omega)(1-\omega^2)\}^2$

$= (1-\omega^2-\omega+\omega^3)^2 = (1-(\omega^2+\omega)+1)^2$

$\omega^4 = \omega$ as $3\,|\,4\,|\,1$, 3, 1

$\omega^8 = \omega^2$ $3\,|\,8\,|\,2$

$= \{2(-1)\}^2$

$= 3^2 = 9$ (Ans)

As $1 + w + w^2 = 0$
$\Rightarrow w + w^2 = -1$

2) Prove that $(2-w)(2-w^2)(2-w^{10})(2-w^{11}) = 49$

Ans

$L.H.S. = (2-w)(2-w^2)(2-w^{10})(2-w^{11})$

$= (2-w)(2-w^2)(2-w)(2-w^2)$

{ as $3 \,|\, 10 \,|\, 3$ (quotient 9, remainder 1) $\Rightarrow w^{10} = w$

and $3 \,|\, 11 \,|\, 3$ (remainder 2) $\Rightarrow w^{11} = w^2$ }

$= (2-w)^2 (2-w^2)^2$

$= \{(2-w)(2-w^2)\}^2 = (4 - 2w^2 - 2w + w^3)^2$

$= (4 - 2(w^2+w) + 1)^2 = (4 - 2(-1) + 1)^2$

{ $1 + w + w^2 = 0 \Rightarrow w + w^2 = -1$ }

$= 7^2 = 49$ (Proved)

3) Prove that $(1 - w + w^2)^7 + (1 + w - w^2)^7 = 128$

Ans

$LHS\ (1 - w + w^2)^7 + (1 + w - w^2)^7$ { use $1 + w + w^2 = 0$ }

$= (1 + w^2 - w)^7 + (1 + w - w^2)^7$

$= (-w - w)^7 + (-w^2 - w^2)^7$

$= (-2w)^7 + (-2w^2)^7 = (-2)^7 \{w^7 + (w^2)^7\}$

$= (-128)(w^7 + w^{14}) = (-128)(w + w^2) = (-128)(-1)$

$= 128$ (Proved)

De Moivre's Theorem

Statement: If $n \in Q$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

E.g. → $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$(\cos\theta + i\sin\theta)^{-2} = \cos(-2\theta) + i\sin(-2\theta)$

$= \cos 2\theta - i\sin 2\theta$

$(\cos\theta + i\sin\theta)^{2/5} = \cos\frac{2}{5}\theta + i\sin\frac{2}{5}\theta$

Application

N th root of a complex number

"$z^{1/n}$"

Procedure

Step-1 → Write the complex number z in its polar form i.e.

$z = r(\cos\theta + i\sin\theta)$

Then $z^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n}$

Step-2

Then number of the roots is n. You have to use consecutive values of θ in order to find all the roots.

Generalised value of θ is $(2k\pi \pm \theta)$

Then $z = r\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}$

Then $z^{1/n} = r^{1/n}\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}^{1/n}$

$\therefore z^{1/n} = r^{1/n}\left\{\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right\}$

(By De Movire's theorem)

By putting $k = 0, 1, 2, \ldots, n-1$
we get the n roots.

Questions

1. Find the three cube roots of $-1+\sqrt{3}\,i$.

Ans

Here $z = -1+\sqrt{3}\,i$

Now we have to find $z^{1/3}$.

First step

So write z in its polar form.

$r = |z| = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$-1+\sqrt{3}\,i$ is in 2nd Quadrant

$\Rightarrow \theta$ lies in the range $\left(\frac{\pi}{2}, \pi\right)$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3}) = 120° = 120° \times \frac{\pi}{180°}$

$= \frac{2\pi}{3}$

Now generalised value of $\theta = \frac{2\pi}{3} + 2k\pi$

Here $z = r\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}$

$= 2\left(\cos\left(2k\pi+\frac{2\pi}{3}\right) + i\sin\left(2k\pi+\frac{2\pi}{3}\right)\right)$

Now $z^{1/3} = 2^{1/3}\left[\cos\frac{2\pi}{3}(3k+1) + i\sin\frac{2\pi}{3}(3k+1)\right]^{1/3}$

$= 2^{1/3}\left[\cos\frac{2\pi}{9}(3k+1) + i\sin\frac{2\pi}{9}(3k+1)\right]$

(By De Movire's theorem)

Now k varies as 0, 1 and 2

$f = 2^{1/3}\cos\frac{2\pi}{9}(3k+1) + i\sin\frac{2\pi}{9}(3k+1)$

$k=0$

1st root $= f_1 = 2^{1/3}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)$

$k=1$

2nd root $= f_2 = 2^{1/3}\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right)$

$= 2^{1/3}\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right)$

$k=2$

3rd root $= f_3 = 2^{1/3}\left(\cos\frac{2\pi}{9}(3\times 2+1) + i\sin\frac{2\pi}{9}(3\times 2+1)\right)$

$= 2^{1/3}\left(\cos\frac{2\pi}{9}(7) + i\sin\frac{2\pi}{9}(7)\right)$

$= 2^{1/3}\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right)$

$= 2^{1/3}\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right)$

2. Solve $z^4 = i$.

Ans

$z^4 = i \Rightarrow z = i^{1/4}$

First we have to write i in polar form

i is on +ve Y axis.

So for i, $r = 1$ and

$\theta = \frac{\pi}{2}$.

$i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= \left[\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)\right]$ (generalised value of θ)

Then $i = \left(\cos\frac{\pi}{2}(4k+1) + i\sin\frac{\pi}{2}(4k+1)\right)$

now $z = i^{1/4} = \left[\cos \frac{\pi}{2}(4k+1) + i \sin \frac{\pi}{2}(4k+1)\right]^{1/4}$

$= \left[\cos \frac{\pi}{8}(4k+1) + i \sin \frac{\pi}{8}(4k+1)\right]$

K varies from 0 to 3

Roots are

$k=0$ $z_1 = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$

$k=1$ $z_2 = \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8}$

$k=2$ $z_3 = \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}$

$k=3$ $z_4 = \cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8}$

Some Important Questions

Q. If $x + \frac{1}{x} = 2\cos\theta$, then show that

(i) $x^n + \frac{1}{x^n} = 2\cos n\theta$

(ii) $x^n - \frac{1}{x^n} = \pm 2 i \sin n\theta$

Proof Given

$x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow x^2 + 1 = 2x\cos\theta$

$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4\times1\times1}}{2\times1}$

[illegible]

$\Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \frac{2\cos\theta \pm \sqrt{4}\sqrt{\cos^2\theta - 1}}{2}$

$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$

$= \cos\theta \pm \sqrt{-1}\sqrt{\sin^2\theta}$

$= \cos\theta \pm i\sin\theta$

Proof of (i)

If $x = \cos\theta + i\sin\theta$

Then $x^n + \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + \frac{1}{(\cos\theta + i\sin\theta)^n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

~~$= (\cos\theta + i\sin\theta)^n$~~ By applying De Moivre theorem

$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

{as $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$}

$= 2\cos n\theta$

If $x = \cos\theta - i\sin\theta$

$x^n + \frac{1}{x^n} = (\cos\theta - i\sin\theta)^n + \frac{1}{(\cos\theta - i\sin\theta)^n}$

$= (\cos\theta - i\sin\theta)^n + (\cos\theta - i\sin\theta)^{-n}$

$= (\cos(-\theta) + i\sin(-\theta))^n + (\cos(-\theta) + i\sin(-\theta))^{-n}$

$= (\cos(-n\theta) + i\sin(-n\theta))$ ([illegible])

$$= (\cos\psi + i\sin\psi) + (\cos\psi + i\sin\psi)$$

$$= \cos n\psi + i\sin n\psi + \cos(-n\psi) + i\sin(-n\psi)$$

$$= \cos n\psi + i\sin n\psi + \cos n\psi - i\sin n\psi$$

$$= 2\cos n\psi = 2\cos n(-\theta) = 2\cos(n\theta)$$

$$= 2\cos n\theta$$ (Proved Part(i))

(ii) Proof of Part-2

When $x = \cos\theta + i\sin\theta$

$$x^n - \frac{1}{x^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$

$$= 2i\sin n\theta \quad \text{---(1)}$$ {as we have done for Part(i)}

When $x = \cos\theta - i\sin\theta$

$$x^n - \frac{1}{x^n} = (\cos n\theta - i\sin n\theta) - (\cos n\theta + i\sin n\theta)$$

$$= -2i\sin n\theta \quad \text{---(2)}$$

Hence from (1) and (2)

$$x^n - \frac{1}{x^n} = \pm 2i\sin n\theta$$

Q2. Show that $\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n$

$$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

Ans

$$L.H.S = \left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n$$

$$= \left(\frac{1+\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)}{1+\cos(\frac{\pi}{2}-\theta) - i\sin(\frac{\pi}{2}-\theta)}\right)^n$$

{As $\sin\theta + i\cos\theta$ is not in complex number form so we have to convert it into $\cos t + i\sin t$ form. For this reason above step is done} Put $\frac{\pi}{2} - \theta = t$

we have

$$= \left(\frac{1+\cos t + i\sin t}{1+\cos t - i\sin t}\right)^n$$

{Let $z = \cos t + i\sin t$. Then $\frac{1}{z} = z^{-1} = (\cos t + i\sin t)^{-1} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$}

$$= \left(\frac{1+z}{1+\frac{1}{z}}\right)^n = \left(\frac{1+z}{\frac{z+1}{z}}\right)^n$$

$$= z^n$$

$$= (\cos t + i\sin t)^n$$

$$= \cos nt + i\sin nt$$ {By De Moivre's theorem}

$$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$$ (Proved)

Q-3 → Given $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$

Then Prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

A. Let $z_1 = \cos\alpha + i\sin\alpha$

$z_2 = \cos\beta + i\sin\beta$ and $z_3 = \cos\gamma + i\sin\gamma$

$\Rightarrow z_1 = 0, \quad z_2 = 0, \quad z_3 = 0$

$\Rightarrow z_1 + z_2 + z_3 = 0$

We know that

$z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3$

$= (z_1+z_2+z_3)(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1)$

$\{ a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ca) \}$

$\Rightarrow z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 = 0 \quad \{ \text{as } z_1+z_2+z_3 = 0 \}$

$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma) = 3\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$

$\left\{ \begin{array}{l} z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \\ \text{Then } z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ \text{(Verify by yourself)} \end{array} \right.$

$\Rightarrow (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3\cos(\alpha+\beta+\gamma) + i\,3\sin(\alpha+\beta+\gamma)$

Equating Real and Imaginary parts we get the result (Proved)

Questions for Practice

1. Obtain square roots of following complex numbers
   (i) $3+4i$
   (ii) $7-24i$
   (iii) $-5+12\sqrt{-1}$

2. Solve $z^3 = 1+i$

3. If α and β are roots of $x^2-2x+4=0$, then show that
$$\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$$

4. For a +ve integer n, show that
$$(1+\sqrt{3}i)^n + (1-\sqrt{3}i)^n = 2^{n+1}\cos\frac{n\pi}{3}$$

(Hints No (3) and (4) are same questions but asked in different way)

5. Show that $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

6. Prove that $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$

(Hints → write denominator as $\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)$ and then apply De Movire's theorem)

7. Find all values of $(1+i)^{1/5}$

8. Prove that $(x-y)(x\omega-y)(x\omega^2-y) = x^3-y^3$

9. Find $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega+\omega^2)(1-\omega^2+\omega)\ldots$ ... $2n$ factors.

Laplace Transforms

Definition

Let $f(t)$ be a function of real variable $t > 0$, then the function $F(s)$ given by $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ is called laplace transform of $f(t)$ provided that $F(s)$ exists.

Mathematically

$$\text{Laplace transform of } f(t) = L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Existence of Laplace Transform

As laplace transform is defined by an improper integral, it may exis[t] or may not.

So, the existency of the improper integral is given by following theorem.

Theorem

The laplace transform of $f(t)$, t[…] exists for all $s > \alpha$, if following conditions are satisfied.

(1) $f(t)$ is a continuous function on every finite interval of $t > 0$.

(2) $|f(t)| \le M e^{\alpha t}$ for some constan[ts] α and M.

Linearity Property

If α and β are any constants

Then $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

Formulas

1. $\mathcal{L}\{K\} = \frac{K}{s}$, $s>0$ {where K is a constant}

2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s>0$ {If $n = 0, 1, 2, \cdots$}

3. $\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ $s>0$ {If $n \in R$}

($\Gamma(n)$ = Gamma n, The gamma-function will be discussed at the end of this topic

→ Only use the formulas $\Gamma(n)=n!$ when $n \in N$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ($\Gamma(0), \Gamma(-1)$ doesn't exist)

4. $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $s>a$

5. $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$ $s>0$

6. $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$ $s>0$

7. $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$ $s>|a|$

8. $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$, $s>|a|$

Derivation of Formula

1. $\mathcal{L}(K) = \int_0^\infty e^{-st} K \, dt$ (Here $f(t)=K$)

$= K\left[\frac{e^{-st}}{-s}\right]_0^\infty$ when $t\to\infty$, $e^{-st}\to 0$ (for $s>0$)

$= \frac{-K}{s}[0 - e^0]$

$= \frac{-K}{s}(-1) = \frac{K}{s}$ ($s>0$)

2. $\mathcal{L}(t) = \int_0^\infty e^{-st} \cdot t \, dt$ (Integration by parts)

$= \left[\frac{e^{-st}}{-s} \cdot t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 1 \cdot dt$

$= -\frac{1}{s}\left[t e^{-st}\right]_0^\infty + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= -\frac{1}{s}\left[0 - 0 \cdot e^{-0}\right] - \frac{1}{s^2}\left[0 - e^0\right]$

($t\to\infty \Rightarrow e^{-st}\to 0$ for $s>0$)

$= 0 - \frac{1}{s^2}(-1) = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$ (for $s>0$)

3. $\mathcal{L}(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$

$= \left[\frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos at + a\sin at\right]\right]_0^\infty$

{as $\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$}

$= \frac{1}{s^2+a^2}\left[e^{-st}\left[-s\cos at + a\sin at\right]\right]_0^\infty$

{ When $t \to \infty$
cos at and sin at are finite
$e^{-st} \to 0$ for $s>0$
$\Rightarrow e^{-st}(-s\cos at + a\sin at) \to 0$ }

$= \frac{1}{s^2+a^2}\left[0 - e^{-0}(-s\cos 0 + a\sin 0)\right]$

$= \frac{1}{s^2+a^2}\left[0 - 1(-s\cdot 1 + 0)\right]$

$= \frac{s}{s^2+a^2}$ (Ans)

$\sinh at$ = sine hyperbolic function $= \frac{e^{at} - e^{-at}}{2}$

$\cosh at = \frac{e^{at} + e^{-at}}{2}$

4. $L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right)$

$= \frac{1}{2}\int_0^\infty e^{-st}(e^{at} - e^{-at})\,dt$

$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right]$

$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$

$= \frac{1}{2}\left[\frac{e^{-(s+a)t}}{s+a} - \frac{e^{-(s-a)t}}{s-a}\right]_0^\infty$

{ when $t \to \infty$ $e^{-(s+a)t} \to 0$ for $s+a>0 \Rightarrow s>-a$
$e^{-(s-a)t} \to 0$ for $s-a>0 \Rightarrow s>a$
Combining both $s > \pm a \Rightarrow s > |a|$ }

$= \frac{1}{2}\left[0 - \left(\frac{e^{-0}}{s+a} - \frac{e^{-0}}{s-a}\right)\right]$

$= \frac{1}{2}\left[-\left(\frac{1}{s+a} - \frac{1}{s-a}\right)\right]$ for $s>|a|$

$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right)$

$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$ (for $s>|a|$)

You can deduce all other formulas by yourself. Try these.

Problems

Find the laplace transform of following

1) $5 + 3t^2 - 4t^3 - 3e^{-2t}$
2) $4\sin 3t - 2\cosh 2t$
3) $7\,3e^{t} - 3t^2 - \sinh t$
4) $4\sqrt{t}$
5) $t^{-1/2}$
6) $3t^{3/2} - \frac{1}{t^{3/2}}$
7) $2\cos 2t - 4\sinh 5t$

(1) $\mathcal{L}(5+3t^2-4t^3-3e^{-2t})$

$= \mathcal{L}(5) + 3\mathcal{L}(t^2) - 4\mathcal{L}(t^3) - 3\mathcal{L}(e^{-2t})$

$= \frac{5}{s} + 3 \cdot \frac{2!}{s^{2+1}} - 4 \cdot \frac{3!}{s^{3+1}} - 3\frac{1}{s-(-2)}$

$= \frac{5}{s} + \frac{3\times 2\times 1}{s^3} - \frac{4\times 3\times 2\times 1}{s^4} - \frac{3}{s+2}$

$= \frac{5}{s} + \frac{6}{s^3} - \frac{24}{s^4} - \frac{3}{s+2}$

2) $\mathcal{L}(4\sin 3t - 2\cosh 2t)$

$= 4\mathcal{L}(\sin 3t) - 2\mathcal{L}(\cosh 2t)$

$= 4\left(\frac{3}{s^2+3^2}\right) - 2\left(\frac{s}{s^2-2^2}\right)$

$= \frac{12}{s^2+9} - \frac{2s}{s^2-4}$

3) $\mathcal{L}(73e^t - 3t^2 - \sinh t)$

$= \mathcal{L}(73e^t) - 3\mathcal{L}(t^2) - \mathcal{L}(\sinh t)$

$= 73\mathcal{L}(e^t) - 3\mathcal{L}(t^2) - \mathcal{L}(\sinh t)$

$= \frac{73}{s-1} - \frac{3\times 2}{s^3} - \frac{1}{s^2-1^2}$

$= \frac{73}{s-1} - \frac{6}{s^3} - \frac{1}{s^2-1}$

4) $\mathcal{L}(4\sqrt{t}) = 4\mathcal{L}(t^{1/2}) = 4\frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}}$

$= \frac{4\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{4\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} = \frac{2\sqrt{\pi}}{s^{3/2}}$

$\left\{ \begin{array}{l} \Gamma(n+1) = n\Gamma(n) \\ \Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) \end{array} \right\}$

5) $\mathcal{L}(t^{-1/2}) = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

6) $\mathcal{L}\left(3t^{3/2} - \frac{1}{t^{3/2}}\right) = 3\mathcal{L}(t^{3/2}) - \mathcal{L}(t^{-3/2})$

$= 3\frac{\Gamma(\frac{5}{2})}{s^{5/2}} - \frac{\Gamma(-\frac{3}{2}+1)}{s^{-3/2+1}}$

$= 3\frac{\frac{3}{2}\Gamma(\frac{3}{2})}{s^{5/2}} - \frac{\Gamma(-\frac{1}{2})}{s^{-1/2}}$

$\left\{ \begin{array}{l} \Gamma(n+1) = n\Gamma(n) \\ \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \end{array} \right.$

$= \frac{9}{2s^{5/2}}\frac{1}{2}\Gamma(\frac{1}{2}) - s^{1/2}\frac{\Gamma(-\frac{1}{2}+1)}{(-\frac{1}{2})}$

$= \frac{9}{4s^{5/2}}\sqrt{\pi} + 2\sqrt{s}\,\Gamma(\frac{1}{2})$

$= \frac{9\sqrt{\pi}}{4s^{5/2}} + 2\sqrt{s}\sqrt{\pi}$ (Ans)

(7) $\mathcal{L}(2\cos 2t - 4\sinh 5t)$

$= 2\mathcal{L}(\cos 2t) - 4\mathcal{L}(\sinh 5t)$

$= 2\left(\frac{s}{s^2+4}\right) - 4\left(\frac{5}{s^2-25}\right)$

$= \frac{2s}{s^2+4} - \frac{20}{s^2-25}$

Q. Evaluate $L(\sin 2t \cdot \cos 3t)$

Ans

$$L(\sin 2t \cdot \cos 3t) = L\left(\frac{\sin(3t+2t) - \sin(3t-2t)}{2}\right)$$

$(\cos A \sin B)$

{ Use formula $(A > B)$

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$ }

$$= \frac{1}{2} L\{\sin 5t - \sin t\}$$

$$= \frac{1}{2}\left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right)$$

$$= \frac{1}{2}\left(\frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)}\right)$$

$$= \frac{1}{2}\frac{4s^2-20}{(s^2+25)(s^2+1)} = \frac{2(s^2-5)}{(s^2+25)(s^2+1)}$$

Q. Find $L\{\cos(at+b)\}$

Ans

$$L\{\cos(at+b)\} = L(\cos at \cdot \cos b - \sin at \sin b)$$

$$= \cos b\, L(\cos at) - \sin b\, L(\sin at)$$

$$= \cos b\left(\frac{s}{s^2+a^2}\right) - \sin b\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{s\cos b - a\sin b}{s^2+a^2}$$

Q. Find $L(\sin^3 2t)$

Ans → $L(\sin^3 2t) = L\left\{\frac{1}{4}(3\sin 2t - \sin 6t)\right\}$

(As we know

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Put $\theta = 2t$ in above)

$\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\Rightarrow \sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$

$$= \frac{1}{4} L(3\sin 2t - \sin 6t)$$

$$= \frac{1}{4}\left[3\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$= \frac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{3}{2}\left[\frac{32}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$

Q. $L(\cos^2 5t)$

Ans → $L(\cos^2 5t) = L\left(\frac{1+\cos 10t}{2}\right)$

{ $\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

Put $\theta = 5t$ in above }

$$= \frac{1}{2} L(1 + \cos 10t)$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+100}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+100+s^2}{s(s^2+100)}\right] = \frac{2s^2+100}{2s(s^2+100)} = \frac{s^2+50}{s(s^2+100)}$$

Q. Find laplace transform of $f(t)$

$$f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

Ans

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} e^t\,dt + \int_1^\infty e^{-st} \cdot 0\,dt$$

$$= \int_0^1 e^{(1-s)t}\, dt + 0$$

$$= \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1 + 0$$

$$= \frac{1}{1-s}\left[e^{(1-s)1} - e^{(1-s)\cdot 0}\right]$$

$$= \frac{1}{(1-s)}\left(e^{(1-s)} - 1\right) \quad \text{(Ans)}$$

Evaluate $L\{f(t)\}$, where

$$f(t) = \begin{cases} t & \text{when } 0 < t < 4 \\ 5 & \text{when } t > 4 \end{cases}$$

Ans

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^4 e^{-st}\cdot t\, dt + \int_4^\infty e^{-st}\cdot 5\, dt$$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^4 - \int_0^4 \frac{e^{-st}}{-s}\cdot 1\, dt + 5\int_4^\infty e^{-st}\, dt$$

$$= -\frac{1}{s}\left[t e^{-st}\right]_0^4 + \frac{1}{s}\int_0^4 e^{-st}\, dt + 5\int_4^\infty e^{-st}\, dt$$

$$= -\frac{1}{s}\left(4e^{-4s} - 0\right) + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

{when as $t \to \infty \Rightarrow e^{-st} \to 0$ for $s > 0$}

$$= -\frac{1}{s}4e^{-4s} - \frac{1}{s^2}\left[e^{-4s} - 1\right] - \frac{5}{s}\left[0 - e^{-4s}\right]$$

$$= -\frac{1}{s}4e^{-4s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + \frac{5}{s}e^{-4s}$$

$$= \frac{1}{s^2}\left[-4se^{-4s} - e^{-4s} + 1 + 5se^{-4s}\right]$$

$$= \frac{1}{s^2}\left[1 + se^{-4s} - e^{-4s}\right]$$

$$= \frac{1}{s^2}\left[1 + e^{-4s}(s-1)\right] \quad \text{(Ans)}$$

First shifting theorem

If $L\, f(t) = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$ ($s \to s-a$)

Ex:

$$L(e^{2t}\sin 2t) = ?$$

$$L(\sin 2t) = \frac{2}{s^2+4} = F(s)$$

By shifting theorem

$$L(e^{2t}\sin 2t) = F(s-a)$$

{where $s \to s-a$, here $a = 2$}

$$= \frac{2}{(s-2)^2+4}$$

Example

$$L(e^{-t}t^2) = ?$$

Ans

$$L(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$L(e^{-t}t^2) = \frac{2}{(s+1)^3}$$

{here $a = -1$, Putting $s \to s-(-1) = s+1$}

Transform of $t^n f(t)$

If $L\{f(t)\} = F(s)$

Then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$

where $n \in N$

E.g. → $L(t\sin t) = ?$

Ans. $L(\sin t) = \frac{1}{s^2+1}$

$L(t\sin t) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= -\left(-\frac{2s}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$

Transform of $\frac{1}{t} f(t)$

If $L f(t) = F(s)$, then

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Example

Find $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

$L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$

$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$

$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$

$= 0 - \log\left(\frac{s+a}{s+b}\right)$

$= \log\left(\frac{s+a}{s+b}\right)^{-1}$

$= \log\left(\frac{s+b}{s+a}\right)$ (Ans)

when $s \to \infty$

$\lim_{s\to\infty} \log\left(\frac{s+a}{s+b}\right)$

$= \log \lim_{s\to\infty}\left(\frac{s+a}{s+b}\right)$

$= \log \lim_{s\to\infty}\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)$

(divide numerator and denominator by s)

$= \log\left(\frac{1+0}{1+0}\right) = \log 1$

$= 0$

Problems

Evaluate the laplace transform of following function

1) $(t+2)^2 e^t$ (2) $t^2 e^t \sin t$

(3) $\frac{\sin^2 t}{t}$ (4) $\frac{\cos 2t - \cos 3t}{t}$

5) $e^{-3t} \sin 5t \sin 3t$ (6) $\sinh 3t \cdot \cos^2 t$

(7) $t e^{-t} \sin 4t$ (8) $e^{-4t} \frac{\sin 3t}{t}$

1) $L\{(t+2)^2 e^t\}$

$L((t+2)^2) = L(t^2 + 4t + 4) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

$L\{(t+2)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}$

(Replace s by $s-1$)

(2) $\mathcal{L}(t^2 e^{t} \sin t)$

Ans

$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$

$\mathcal{L}(t^2 \sin t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = 1 \cdot \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$

$= \frac{d}{ds}\left\{-\frac{1 \cdot 2s}{(s^2+1)^2}\right\} = -2 \frac{d}{ds}\left\{\frac{s}{(s^2+1)^2}\right\}$

$= -2\left\{\frac{1 \cdot (s^2+1)^2 - s \cdot 2(s^2+1) \cdot 2s}{(s^2+1)^4}\right\}$

$= -2 \frac{(s^2+1)\{s^2+1 - 2s \cdot 2s\}}{(s^2+1)^4}$

$= \frac{-2}{(s^2+1)^3}\{s^2+1-4s^2\} = \frac{-2(1-3s^2)}{(s^2+1)^3}$

$= \frac{2(3s^2-1)}{(s^2+1)^3}$

Now $\mathcal{L}(t^2 e^{t} \sin t) = \frac{2(3(s-1)^2-1)}{\{(s-1)^2+1\}^3}$ (1st shifting theorem)

$= \frac{2\{3(s^2-2s+1)-1\}}{(s^2-2s+1+1)^3}$

$= \frac{2\{3s^2-6s+3-1\}}{(s^2-2s+2)^3}$

$= \frac{2(3s^2-6s+2)}{(s^2-2s+2)^3}$ (Ans)

(3) $\mathcal{L}\left(\frac{\sin^2 t}{t}\right)$

Ans

$\mathcal{L}(\sin^2 t) = \mathcal{L}\left(\frac{1-\cos 2t}{2}\right)$ $\{\cos 2t = 1 - 2\sin^2 t$

$= \frac{1}{2}\mathcal{L}(1-\cos 2t)$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

Now $\mathcal{L}\left(\frac{\sin^2 t}{t}\right) = \int_s^{\infty} \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^{\infty}$

$\left\{ \int \frac{s}{s^2+4} ds = \int \frac{du}{2u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(s^2+4) \right.$ $\quad s^2+4 = u \Rightarrow 2s\,ds = du$

$= \frac{1}{2}\left[\ln s - \ln\sqrt{s^2+4}\right]_s^{\infty}$

$= \frac{1}{2}\left[\ln \frac{s}{\sqrt{s^2+4}}\right]_s^{\infty}$

$\left\{ \lim_{s\to\infty} \ln \frac{s}{\sqrt{s^2+4}} = \ln \lim_{s\to\infty} \frac{s}{\sqrt{s^2+4}} = \ln \lim_{s\to\infty} \frac{\frac{s}{s}}{\sqrt{\frac{s^2}{s^2}+\frac{4}{s^2}}} = \ln \lim_{s\to\infty} \frac{1}{\sqrt{1+\frac{4}{s^2}}} = \ln \frac{1}{\sqrt{1+0}} = \ln 1 = 0 \right.$

$= \frac{1}{2}\left(0 - \ln \frac{s}{\sqrt{s^2+4}}\right)$

$= \frac{1}{2}\ln\left(\frac{\sqrt{s^2+4}}{s}\right)$

$= \frac{1}{2}\ln\sqrt{\frac{s^2+4}{s^2}}$

$= \frac{1}{2} \times \frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right)$

$= \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$ (Ans)

4) $L\left(\frac{\cos 2t - \cos 3t}{t}\right)$

Ans

$$L(\cos 2t - \cos 3t) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

$$L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \left[\frac{1}{2}\ln(s^2+4) - \frac{1}{2}\ln(s^2+9)\right]_s^\infty$$

$\left\{ \text{As } \int \frac{s}{s^2+4}\, ds = \frac{1}{2}\ln(s^2+4) \right.$

$\left. \int \frac{s}{s^2+9}\, ds = \frac{1}{2}\ln(s^2+9) \right\}$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left\{0 - \ln\left(\frac{s^2+4}{s^2+9}\right)\right\}$$

{As we have done previously $\lim_{s\to\infty} \ln\left(\frac{s^2+4}{s^2+9}\right) = 0$}

$$= \frac{1}{2}\ln\left(\frac{s^2+4}{s^2+9}\right)^{-1} = \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+4}\right) \text{ (Ans)}$$

5) $L(e^{-3t}\sin 5t \sin 3t)$

Ans

$$L(\sin 5t \sin 3t) = \frac{1}{2} L\{\cos(5t-3t) - \cos(5t+3t)\}$$

$\{2\sin A \sin B = \cos(A-B) - \cos(A+B)\}$

$$= \frac{1}{2} L(\cos 2t - \cos 8t)$$

$$= \frac{1}{2}\left(\frac{s}{s^2+4} - \frac{s}{s^2+64}\right) \qquad s \to s-(-3) = s+3$$

$$L(e^{-3t}\sin 5t \sin 3t) = \frac{1}{2}\left[\frac{s+3}{(s+3)^2+4} - \frac{s+3}{(s+3)^2+64}\right]$$

$$= \frac{1}{2}(s+3)\left(\frac{1}{s^2+6s+9+4} - \frac{1}{s^2+6s+9+64}\right)$$

$$= \frac{(s+3)}{2}\left(\frac{1}{s^2+6s+13} - \frac{1}{s^2+6s+73}\right)$$

$$= (s+3)\left(\frac{s^2+6s+73 - s^2-6s-13}{(s^2+6s+13)(s^2+6s+73)}\right)$$

$$= \frac{(s+3)}{2}\frac{60}{(s^2+6s+13)(s^2+6s+73)}$$

$$= \frac{~~60~~ (s+3)\, 30}{(s^2+6s+13)(s^2+6s+73)}$$

6) $L(\sinh 3t \cos^2 t)$

Ans

$$L(\sinh 3t \cos^2 t) = L\left\{\left(\frac{e^{3t} - e^{-3t}}{2}\right)\cos^2 t\right\}$$

$$= \frac{1}{2} L(e^{3t}\cos^2 t - e^{-3t}\cos^2 t) \quad \text{—(1)}$$

$$L(\cos^2 t) = L\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right) \quad \text{—(2)}$$

From (1) and (2)

$$L(\sinh 3t \cos^2 t) = \frac{1}{2} L(e^{3t}\cos^2 t - e^{-3t}\cos^2 t)$$

$$= \frac{1}{2} \times \left[\frac{1}{2}\left\{\frac{1}{s-3} + \frac{s-3}{(s-3)^2+4}\right\} - \frac{1}{2}\left(\frac{1}{s+3} + \frac{s+3}{(s+3)^2+4}\right)\right]$$

$$= \frac{1}{4}\left[\frac{s^2-6s+13 + s^2-6s+9}{(s-3)(s^2-6s+13)} - \frac{s^2+6s+13+s^2+6s+9}{(s+3)(s^2+6s+13)}\right]$$

$$= \frac{1}{4}\left[\frac{2(s^2-6s+11)}{(s-3)(s^2-6s+13)} - \frac{2(s^2+6s+11)}{(s+3)(s^2+6s+13)}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-6s+11}{(s-3)(s^2-6s+13)} - \frac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$$

(Ans)

=) $L(te^{-t}\sin 4t)$

Ans

$$L(\sin 4t) = \frac{4}{s^2+16}$$

$$L(t\sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = -\left(-\frac{4\cdot 2s}{(s^2+16)^2}\right)$$

$$= \frac{8s}{(s^2+16)^2}$$

$$L(te^{-t}\sin 4t) = L\{e^{-t}(t\sin 4t)\}$$

$$= \frac{8(s+1)}{\{(s+1)^2+16\}^2} \qquad \{\text{Shifting theorem}\}$$

$$= \frac{8(s+1)}{(s^2+2s+17)^2}$$

8) $L\left(e^{-4t}\frac{\sin 3t}{t}\right)$

Ans

$$L(\sin 3t) = \frac{3}{s^2+9}$$

$$L\left(\frac{\sin 3t}{t}\right) = \int_s^\infty \frac{3}{s^2+9}\,ds$$

$$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad \left\{\lim_{s\to\infty}\tan^{-1}\left(\frac{s}{3}\right) = \frac{\pi}{2}\right.$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \cot^{-1}\left(\frac{s}{3}\right)$$

Now $L\left(e^{-4t}\frac{\sin 3t}{t}\right) = \cot^{-1}\left(\frac{s+4}{3}\right)$ (Ans)

Practice Problems

1) Find laplace Transform of functions

a) $(1+te^{-t})^3$ b) $e^{-2t}\sinh 3t$

(c) $\cosh^2 at$ (d) $\sin(at+b)$

(e) $f(t) = \begin{cases} 1 & \text{when } 0 \le t \le 2 \\ t & \text{when } t > 2 \end{cases}$

2) Find laplace Transform of following functions

(a) $e^{-t}\sin^2 t$ (b) $e^{-4t}\cos t\sin 2t$

(c) $te^{-t}\sin 4t$ (d) $\frac{e^{2t}-\cos 3t}{t}$

(e) $\frac{1-\cos t}{t}$ (f) $\cosh at\sin at$

(g) $t^2\sin 2t$ (h) $t\sin^2 t$

(i) Given $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$ then show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$

Inverse Laplace Transform

If $\mathcal{L}(f(t)) = F(s)$ then Laplace inverse of $F(s)$ is $f(t)$ denoted by

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Formulas

1. $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

2. $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

3. $\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$ (when $n \in N$)

4. $\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma(n)}$ (when n is not a natural number)

5. $\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

6. $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

7. $\mathcal{L}^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

8. $\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

Property

$$\mathcal{L}^{-1}\left(\alpha F(s) + \beta G(s)\right) = \alpha\, \mathcal{L}^{-1}\{F(s)\} + \beta\, \mathcal{L}^{-1}(G(s))$$

E.g.

1) $\mathcal{L}^{-1}\left(\frac{3}{s}\right) = 3\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 3\times 1 = 3$

2) $\mathcal{L}^{-1}\left(\frac{2}{s-4}\right) = 2\,\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) = 2e^{4t}$

3) $\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$

4) $\mathcal{L}^{-1}\left(\frac{1}{s^2} + 2s^{-3/2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 2\,\mathcal{L}^{-1}\left(s^{-3/2}\right)$

$= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 2\,\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right)$

$= \frac{t^{2-1}}{(2-1)!} + 2\frac{t^{3/2-1}}{\Gamma\left(\frac{3}{2}\right)}$

$= \frac{t}{1!} + \frac{2t^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$ ~~$= t + \frac{4\sqrt{t}}{\sqrt{\pi}}$~~

$= t + 2\times 2\frac{\sqrt{t}}{\sqrt{\pi}}$

$= t + 4\sqrt{\frac{t}{\pi}}$ (Ans)

5) $\mathcal{L}^{-1}\left(\frac{3}{s^2+2} - \frac{s}{s^2-9}\right)$

$= \mathcal{L}^{-1}\left(\frac{3}{s^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2-3^2}\right)$

$= 3\,\mathcal{L}^{-1}\left(\frac{1}{s^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2-3^2}\right)$

$= 3\frac{1}{2}\sin 2t - \cosh 3t = \frac{3}{2}\sin 2t - \cosh 3t$

$$L^{-1}\left(\frac{3s-2}{s^2+8}\right) = L^{-1}\left(\frac{3s}{s^2+8}\right) - 2L^{-1}\left(\frac{1}{s^2+8}\right)$$

$$= 3L^{-1}\left(\frac{s}{s^2+(\sqrt{8})^2}\right) - 2L^{-1}\left(\frac{1}{s^2+(\sqrt{8})^2}\right)$$

$$= 3\cos\sqrt{8}t - \frac{2}{\sqrt{8}}\sin\sqrt{8}t$$

$$= 3\cos\sqrt{8}t - \frac{2}{2\sqrt{2}}\sin\sqrt{8}t$$

$$= 3\cos\sqrt{8}t - \frac{1}{\sqrt{2}}\sin\sqrt{8}t$$

$$\Rightarrow L^{-1}\left(\frac{5s-3}{16-s^2}\right) = L^{-1}\left(\frac{3-5s}{s^2-16}\right)$$

$$= L^{-1}\left(\frac{3}{s^2-16}\right) - L^{-1}\left(\frac{5s}{s^2-16}\right)$$

$$= 3L^{-1}\left(\frac{1}{s^2-4^2}\right) - 5L^{-1}\left(\frac{s}{s^2-4^2}\right)$$

$$= 3\,\frac{1}{4}\sinh 4t - 5\cosh 4t$$

$$= \frac{3}{4}\sinh 4t - 5\cosh 4t$$

Formula

$$L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}$$

When $(s-a)$ is present in place of s then we use ~~Formd~~ formula
Replace $s-a$ by s and multiply L^{-1} with e^{at}.

Examples

1. $L^{-1}\left(\frac{s-3}{(s-3)^2+4^2}\right)$ (Here $s-3$ is in place of s)

$$= e^{3t}L^{-1}\left(\frac{s}{s^2+4^2}\right) \quad \{\text{Replace } s-3 \text{ by } s\}$$

$$= e^{3t}\cos 4t$$

2. $$L^{-1}\left\{\frac{1}{(s-2)^2}\right\} = e^{2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{2t}\frac{t^{2-1}}{(2-1)!} = e^{2t}\frac{t}{1!} = te^{2t}$$

(3) $$L^{-1}\left(\frac{3}{(s+1)^2-16}\right) = 3L^{-1}\left(\frac{1}{(s+1)^2-4^2}\right)$$

$$= 3e^{-t}L^{-1}\left(\frac{1}{s^2-4^2}\right)$$

{Here $s+1$ i.e. $(s-(-1))$ is in place of s. So apply formula as applied here}

$$= 3e^{-t}\frac{1}{4}\sinh 4t$$

$$= \frac{3}{4}e^{-t}\sinh 4t$$

4. $$L^{-1}\left(\frac{7}{s^2+10s+20}\right) = L^{-1}\left(\frac{7}{s^2+2\cdot s\cdot 5+5^2-5^2+20}\right)$$

$$= L^{-1}\left(\frac{7}{(s+5)^2-5}\right) = 7L^{-1}\left(\frac{1}{(s+5)^2-(\sqrt{5})^2}\right)$$

Here $s+5$ is in place of s i.e. $s-(-5)$ is in place of s

$$= 7e^{-5t}L^{-1}\left(\frac{1}{s^2-(\sqrt{5})^2}\right)$$

$$= 7e^{-5t}\frac{1}{\sqrt{5}}\sinh\sqrt{5}t = \frac{7}{\sqrt{5}}e^{-5t}\sinh\sqrt{5}t$$

5) $\mathcal{L}^{-1}\left[\dfrac{s^2+s+2}{s^{3/2}}\right]$

$$= \mathcal{L}^{-1}\left(\frac{s^2}{s^{3/2}} + \frac{s}{s^{3/2}} + \frac{2}{s^{3/2}}\right)$$

$$= \mathcal{L}^{-1}\left(s^{1/2} + \frac{1}{s^{1/2}} + \frac{2}{s^{3/2}}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^{-1/2}} + \frac{1}{s^{1/2}} + \frac{2}{s^{3/2}}\right)$$

$$= \frac{t^{-\frac{1}{2}-1}}{\Gamma\left(-\frac{1}{2}\right)} + \frac{t^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)} + \frac{2t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{t^{-3/2}}{\frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)}} + \frac{t^{-1/2}}{\sqrt{\pi}} + \frac{2t^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{\pi}\, t^{3/2}} + \frac{1}{\sqrt{\pi}\sqrt{t}} + 4\frac{\sqrt{t}}{\sqrt{\pi}}$$

$$= -\frac{1}{2\sqrt{\pi}\, t^{3/2}} + \frac{1}{\sqrt{t\pi}} + 4\sqrt{\frac{t}{\pi}}$$

6) $\mathcal{L}^{-1}\left(\dfrac{s+2}{s^2-4s+13}\right) = \mathcal{L}^{-1}\left(\dfrac{s+2}{s^2-2\cdot s\cdot 2+2^2-2^2+13}\right)$

$$= \mathcal{L}^{-1}\left(\frac{s+2}{(s-2)^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s-2+2+2}{(s-2)^2+9}\right) = \mathcal{L}^{-1}\left(\frac{(s-2)+4}{(s-2)^2+9}\right)$$

$s-2$ is inplace of s

$$= e^{2t}\,\mathcal{L}^{-1}\left(\frac{s+4}{s^2+3^2}\right) = e^{2t}\left\{\mathcal{L}^{-1}\left(\frac{s}{s^2+3^2}\right) + \mathcal{L}^{-1}\left(\frac{4}{s^2+3^2}\right)\right\}$$

$$= e^{2t}\left(\cos 3t + \frac{4}{3}\sin 3t\right) \quad \text{(Ans)}$$

Partial Fraction

Algebraic functions $\frac{P(x)}{Q(x)}$ resolved into proper fractions called partial fractions.

Proper fraction → $\frac{P(x)}{Q(x)}$ where degree of $P(x)$ is less than $Q(x)$

Techniques

(1) If $Q(x) = (x-a)(x-b)(x-c)\ldots\ldots$ partial fractions.

Then $\dfrac{P(x)}{Q(x)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \ldots\ldots$

where $A, B, C \ldots\ldots$ are constants

(2) If $Q(x) = (x-a)^2(x-b)(x-c)$

Then $\dfrac{P(x)}{Q(x)} = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b} + \dfrac{D}{x-c}$

3) If $Q(x) = (x^2+\alpha)(x^2+\beta)(x-\gamma)$

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{x^2+\alpha} + \frac{Cx+D}{x^2+\beta} + \frac{E}{x-\gamma}$$

4) If $Q(x) = (x^2+\alpha)^2(x-\beta)$

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{x^2+\alpha} + \frac{Cx+D}{(x^2+\alpha)^2} + \frac{E}{(x-\beta)}$$

⑦

Q. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$

Ans

First we have to find the partial fraction of $\dfrac{s^2+s-2}{s(s+3)(s-2)}$. Then we can evalute directly by applying formulas.

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$\Rightarrow \frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A(s+3)(s-2) + Bs(s-2) + Cs(s+3)}{s(s+3)(s-2)}$$

$$\Rightarrow s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3) \quad \text{--- (1)}$$

Now we have to evalute A, B, C.

Putting $s = 0$ in (1)

$$\Rightarrow -2 = A \times 3 \times (-2) + 0 + 0$$

$$\Rightarrow -2 = -6A \quad \Rightarrow \boxed{A = \frac{-2}{-6} = \frac{1}{3}} \quad \text{--- (2)}$$

Putting $s = 2$ in (1)

$$2^2 + 2 - 2 = 0 + 0 + C \times 2(2+3)$$

$$\Rightarrow 4 = 10C \quad \Rightarrow \boxed{C = \frac{4}{10} = \frac{2}{5}} \quad \text{--- (3)}$$

Putting $s = -3$ in (1)

$$(-3)^2 + (-3) - 2 = 0 + B(-3)(-3-2) + 0$$

$$\Rightarrow 4 = 15B \quad \Rightarrow \boxed{B = \frac{4}{15}} \quad \text{--- (4)}$$

⑧

From (2), (3), (4) we have

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$= \frac{1}{3}\left(\frac{1}{s}\right) + \frac{4}{15}\left(\frac{1}{s+3}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right)$$

Now $\mathcal{L}^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{3}\left(\dfrac{1}{s}\right) + \dfrac{4}{15}\left(\dfrac{1}{s+3}\right) + \dfrac{2}{5}\left(\dfrac{1}{s-2}\right)\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{4}{15}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \frac{2}{5}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)$$

$$= \frac{1}{3}\cdot 1 + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

Q→ Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2-1)^2}\right\}$

Ans

$$\frac{s}{(s^2-1)^2} = \frac{s}{\{(s+1)(s-1)\}^2} = \frac{s}{(s+1)^2(s-1)^2}$$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2} \quad \text{--- (1)}$$

$$\Rightarrow \frac{s}{(s^2-1)^2} = \frac{~~A(s+1)+B~~\ A(s+1)(s-1)^2 + B(s-1)^2 + C(s+1)^2(s-1) + D(s+1)^2}{(s^2-1)^2}$$

$$\Rightarrow s = A(s+1)(s-1)^2 + B(s-1)^2 + C(s+1)^2(s-1) + D(s+1)^2$$

~~$\Rightarrow s = A(s^2-1) + B(s-1) + C(s^2-1) + D(s-1)$~~

~~$= As^2 - A + Bs - B + Cs^2 - C + Ds - D$~~

~~$= (A+C)s^2 + (B+D)s - (A+B+C+D)$~~

$$\Rightarrow s = (As + A)(s^2 - 2s + 1) + B(s^2 - 2s + 1)$$
$$+ (Cs - C)(s^2 + 2s + 1) + D(s^2 + 2s + 1)$$

$\Rightarrow s = As^3 - 2As^2 + As + As^2 - 2As + A$
$+ Bs^2 - 2Bs + B + Cs^3 + 2Cs^2 + Cs$
$- Cs^2 - 2Cs - C + Ds^2 + 2Ds + D$

$$\Rightarrow s = (A+C)s^3 + (-2A + A + B + 2C - C + D)s^2 + (A - 2A - 2B + C - 2C + 2D)s + (A + B - C + D)$$

Equating co-efficient of s^3, of both side

$\Rightarrow 0 = A + C$

$\Rightarrow \boxed{A = -C}$ ——— (2)

Equating coefficient of s^2, of both side

$0 = (-2A + A + B + 2C - C + D)$

$\Rightarrow -A + B + C + D = 0$ ——— (3)

Equating coefficient of s,

$1 = (A - 2A - 2B + C - 2C + 2D)$

$\Rightarrow \boxed{1 = -A - 2B - C + 2D}$ ——— (4)

Equating constant terms:

$\boxed{A + B - C + D = 0}$ ——— (5)

From (3) and 5

$-A + B + C + D = 0$
$A + B - C + D = 0$
(+) ————————
$2B + 2D = 0$

$\Rightarrow \boxed{B = -D}$ ——— (6)

From (2), (4) and (6)

$1 = -(-C) - 2(-D) - C + 2D$

$\Rightarrow 1 = C + 2D - C + 2D$

$\Rightarrow 4D = 1 \Rightarrow \boxed{D = \frac{1}{4}}$

$\boxed{B = -D = -\frac{1}{4}}$

~~Put the value of B and D in (5)~~

~~$A + \left(-\frac{1}{4}\right) - C + \frac{1}{4} = 0$~~

~~$A - \frac{1}{4} - C + \frac{1}{4}$~~

Put the value of B and D in (3)

$-A - \frac{1}{4} + C + \frac{1}{4} = 0$

$\Rightarrow C - A = 0$

$\Rightarrow C - (-C) = 0$

$\Rightarrow 2C = 0 \Rightarrow \boxed{C = 0}$

$\Rightarrow \boxed{A = -C = 0}$

From (1)

$$\frac{s}{(s^2-1)^2} = -\frac{1}{4}\frac{1}{(s+1)^2} + \frac{1}{4}\frac{1}{(s-1)^2}$$

$$L^{-1}\left\{\frac{s}{(s^2-1)^2}\right\} = -\frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= -\frac{1}{4}e^{-t}L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{4}e^{t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= -\frac{1}{4}e^{-t}\frac{t^{2-1}}{1!} + \frac{1}{4}e^{t}\frac{t^{2-1}}{1!}$$

$$= \frac{t}{4}\left[e^{t} - e^{-t}\right] = \frac{t}{2}\left(\frac{e^{t} - e^{-t}}{2}\right)$$

$$= \frac{t}{2}\sinh t \quad \text{(Ans)}$$

Q.3. $L^{-1}\left(\frac{s^2+6}{(s^2+1)(s^2+4)}\right)$

Ans → $\frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$ — (1)

$\Rightarrow s^2+6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

$\Rightarrow s^2+6 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$

$\Rightarrow s^2+6 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$ — (2)

Equating coefficient of s^3 on both sides

$\Rightarrow A+C=0 \Rightarrow \boxed{A=-C}$ — (3)

Equating coefficient of s^2 on both sides

$\Rightarrow B+D=1 \Rightarrow \boxed{B=1-D}$ — (4)

Equating coefficient of s on both sides.

$\Rightarrow 4A+C=0$

$\Rightarrow 4(-C)+C=0$ {from (3)}

$\Rightarrow -3C=0 \Rightarrow \boxed{C=0}$

$\Rightarrow \boxed{A=0}$

Equating constant terms in (2),

$4B+D=6$

$\Rightarrow 4(1-D)+D=6$

$\Rightarrow 4-4D+D=6$

$\Rightarrow -3D=2$

$\Rightarrow \boxed{D=\frac{-2}{3}}$ $\Rightarrow B = 1-\left(\frac{-2}{3}\right) = 1+\frac{2}{3}$

$\Rightarrow \boxed{B=\frac{5}{3}}$

Putting value of A, B, C and D in (1)

$\frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{\frac{5}{3}}{s^2+1} + \frac{\left(\frac{-2}{3}\right)}{s^2+4}$

Now

$L^{-1}\left\{\frac{s^2+6}{(s^2+1)(s^2+4)}\right\} = \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{2}{3}L^{-1}\left(\frac{1}{s^2+4}\right)$

$= \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{2}{3}L^{-1}\left(\frac{1}{s^2+2^2}\right)$

$= \frac{5}{3}\sin t - \frac{2}{3}\cdot\frac{1}{2}\sin 2t$

$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$ (Ans)

Formulas

(1) If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$ then

$\boxed{L^{-1}\{sF(s)\} = \frac{d}{dt}f(t)}$

(2) If $L^{-1}(F(s)) = f(t) \Rightarrow \boxed{L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt}$

(3) If $L^{-1}[F(s)] = f(t)$, then

$t\,f(t) = L^{-1}\left[-\frac{d}{ds}F(s)\right]$

(4) If $L^{-1}[F(s)] = f(t)$ then,

$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds.$

$\Rightarrow \boxed{f(t)/t = L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}}$

Examples

a 1) $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

Log, $\tan^{-1}$, $\cot^{-1}$ etc functions laplace transformation has no formula So, differentiate them to apply the formula
So we the no (3) formula

Ans
we know if $L^{-1}\{F(s)\} = f(t)$

Then $t f(t) = L^{-1}\left\{-\frac{d}{ds}F(s)\right\}$

Here $F(s) = \log\left(\frac{s+1}{s-1}\right)$

Aim → $f(t) = ?$

Now applying formula

$$t f(t) = L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\}$$

$$= L^{-1}\left\{-\left\{\frac{d}{ds}\left(\log(s+1) - \log(s-1)\right)\right\}\right\}$$

$$= L^{-1}\left(-\left(\frac{1}{s+1} - \frac{1}{s-1}\right)\right)$$

$$= L^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$= e^{t} - e^{-t}$$

$$\Rightarrow f(t) = \frac{e^{t} - e^{-t}}{t} = \frac{2}{t}\left(\frac{e^{t} - e^{-t}}{2}\right)$$

$$= \frac{2}{t}\sinh t$$

2) $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

Ans
$F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

Let $L^{-1}\{F(s)\} = f(t)$

Then $t f(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$$= L^{-1}\left\{-\frac{2}{1+\left(\frac{2}{s^2}\right)^2}\frac{(-2)}{s^3}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{\frac{(s^4+4)}{s^4}\cdot s^3}\right\}$$

$$= 4L^{-1}\left(\frac{s}{s^4+4}\right)$$

$$= 4L^{-1}\left\{\frac{s}{(s^2+2)^2 - 2\cdot s^2\cdot 2}\right\}$$

$\{ a^2 + b^2 = (a+b)^2 - 2ab$

$$= 4L^{-1}\left\{\frac{s}{(s^2+2)^2 - (2s)^2}\right\}$$

$$= 4L^{-1}\left\{\frac{s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$\left[\frac{1}{s^2+2s+2} - \frac{1}{(s^2-2s+2)} = \frac{s^2-2s+2 - s^2-2s-}{(s^2+2s+2)(s^2-2s+}\right.$

$$= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)}\Big]$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+1+1} - \frac{1}{s^2+2s+1+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{t} L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{t}\sin t - e^{-t}\sin t$$

$$= \sin t\,(e^{t}-e^{-t}) = 2\sin t\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$= 2\sin t\,\sinh t$$

3) $L^{-1}\left\{\frac{1}{s(s+2)^3}\right\}$

{ Here $\frac{1}{s}$ is multiplied with $\frac{1}{(s+2)^3}$
So used $L^{-1}\frac{1}{s}F(s)$ formula }

Here $L^{-1}\left\{\frac{1}{(s+2)^3}\right\} = e^{-2t} L^{-1}\left(\frac{1}{s^3}\right)$

$$= e^{-2t}\frac{t^{3-1}}{2!} = \frac{1}{2}t^2 e^{-2t}$$

Hence $L^{-1}\left\{\frac{1}{s}\frac{1}{(s+2)^3}\right\} = \int_0^t \frac{1}{2}t^2 e^{-2t}\,dt$

$$= \frac{1}{2}\left[\left[\frac{e^{-2t}}{-2}t^2\right]_0^t - \int_0^t 2t\frac{e^{-2t}}{-2}dt\right]$$

(11)

$$= \frac{1}{2}\left[\left(\frac{-e^{-2t}t^2}{2} - 0\right) + \int_0^t t e^{-2t}\,dt\right]$$

$$= -\frac{1}{4}t^2 e^{-2t} + \frac{1}{2}\left[\left[t\frac{e^{-2t}}{-2}\right]_0^t - \int_0^t 1\,\frac{e^{-2t}}{-2}dt\right]$$

$$= -\frac{1}{4}t^2 e^{-2t} + \frac{1}{2}\left[\left(-\frac{te^{-2t}}{2} - 0\right) + \frac{1}{2}\left[\frac{e^{-2t}}{-2}\right]_0^t\right]$$

$$= -\frac{1}{4}t^2 e^{-2t} + \frac{1}{2}\left[-\frac{te^{-2t}}{2} - \frac{1}{4}\left(e^{-2t}-1\right)\right]$$

$$= -\frac{1}{4}t^2 e^{-2t} - \frac{1}{4}te^{-2t} - \frac{1}{8}e^{-2t} + \frac{1}{8}$$

$$= \frac{1}{8} - \frac{1}{8}e^{-2t}(2t^2+2t+1) \quad \text{(Ans)}$$

4) $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$ { Apply $L^{-1}\left(\frac{1}{s}F(s)\right)$ formula three times }

Ans

$$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$L^{-1}\left(\frac{1}{s}\frac{1}{s^2+1}\right) = \int_0^t \sin t\,dt = \left[-\cos t\right]_0^t$$

$$= -\left[\cos t - \cos 0\right] = 1-\cos t$$

$$L^{-1}\left\{\frac{1}{s}\cdot\left(\frac{1}{s}\cdot\frac{1}{s^2+1}\right)\right\} = \int_0^t (1-\cos t)\,dt$$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right) = \left[t-\sin t\right]_0^t$$

$$= \left[(t-\sin t) - (0-\sin 0)\right]$$

$$= t - \sin t$$

$$L^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2}\ \frac{1}{s^2+1}\right)\right\} = \int_0^t (t-\sin t)\,dt$$

$$= \left[\frac{t^2}{2} + \cos t\right]_0^t$$

$$= \left(\frac{t^2}{2} + \cos t\right) - (0 + \cos 0)$$

$$= \frac{t^2}{2} + \cos t - 1$$

Q. $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

ns

~~Here $L^{-1}\left(\frac{1}{s^2+a^2}\right) =$~~

Here $\frac{s}{(s^2+a^2)^2}$ is integrable easily.

So let $F(s) = \frac{s}{(s^2+a^2)^2}$

and $L^{-1}\{F(s)\} = f(t)$

Then $\frac{f(t)}{t} = L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$

$$= L^{-1}\left\{\int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds\right\}$$

$\left[\int \frac{s}{(s^2+a^2)^2}\,ds = \frac{1}{2}\int \frac{du}{u^2}\,du \qquad s^2+a^2 = u\right.$

$\Rightarrow 2s\,ds = du$

$\Rightarrow s\,ds = \frac{du}{2}$

$\left.= \frac{1}{2}\left(-\frac{1}{u}\right) = -\frac{1}{2u}\right]$

$$= -\frac{1}{2(s^2+a^2)}$$

$$= L^{-1}\left\{\left[-\frac{1}{2(s^2+a^2)}\right]_s^\infty\right\}$$

$\left(s \to \infty,\ \frac{1}{s^2+a^2} \to 0\right)$

$$= L^{-1}\left\{-\left(0 - \frac{1}{2(s^2+a^2)}\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{2(s^2+a^2)}\right\} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{1}{2a}\sin at$$

Q $L^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\}$

Let $F(s) = \frac{s^2-a^2}{(s^2+a^2)^2}$ and $L^{-1}F(s) = f(t)$

$$\Rightarrow \frac{f(t)}{t} = L^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$$

$$= L^{-1}\left\{\int_s^\infty \frac{s^2-a^2}{(s^2+a^2)^2}\,ds\right\}$$

$$= L^{-1}\left\{\int_s^\infty -\frac{\{(s^2+a^2).1 - s(2s\}}{(s^2+a^2)^2}\,ds\right\}$$

$$= L^{-1}\left\{\int_s^\infty -d\left(\frac{s}{s^2+a^2}\right)\right\}$$

$$= -L^{-1}\left\{\left[\frac{s}{s^2+a^2}\right]_s^\infty\right\} = -L^{-1}\left(0 - \frac{s}{s^2+a^2}\right)$$

$$= L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$\Rightarrow \boxed{f(t) = t\cos at}$$

Long Exercise Book

Q $\mathcal{L}^{-1} \log\left(\frac{s^2+1}{s(s+1)}\right)$ (17)

Ans

Let $F(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$

Let $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$\Rightarrow t f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}$

$= -\mathcal{L}^{-1}\frac{d}{ds}\left\{\log(s^2+1) - \log s(s+1)\right\}$

$= -\mathcal{L}^{-1}\frac{d}{ds}\left\{\log(s^2+1) - \log s - \log(s+1)\right\}$

$= -\mathcal{L}^{-1}\left\{\frac{1 \cdot 2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$

$= -\left\{2\cos t - 1 - e^{-t}\right\}$

$= 1 + e^{-t} - 2\cos t$

$\Rightarrow t f(t) = 1 + e^{-t} - 2\cos t$

$\Rightarrow \boxed{\mathcal{L}^{-1}\log\left(\frac{s^2+1}{s(s+1)}\right) = \frac{1+e^{-t}-2\cos t}{t}}$ (Ans)

Q $\mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+5s+9)}\right\}$

Ans

Let $\frac{5s+3}{(s-1)(s^2+5s+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+5s+9}$ — (1)

s^2+5s+9 cannot factorized as $5^2-4 \cdot 1 \cdot 9 < 0$

$\Rightarrow 5s+3 = A(s^2+5s+9) + (Bs+C)(s-1)$

$= As^2 + 5As + 9A + Bs^2 - Bs + Cs - C$

$= (A+B)s^2 + (5A-B+C)s + (9A-C)$ — (2)

Equating coefficient of s^2

$A+B=0$ — (3)

Equating coefficients of s

$5A-B+C=5$ — (4)

Equating constant terms $\boxed{9A-C=3}$ — (5)

From (3) $A=-B$ ~~and A = 3+C~~

From (5) $C = 9A-3$

using these in (4) $5A-(-A)+(9A-3)=5$

$\Rightarrow 15A-3=5 \Rightarrow \boxed{A=\frac{8}{15}}$

$B = -A = -\frac{8}{15}$

$C = 9\times\frac{8}{15} - 3 = \frac{24}{5} - 3 = \frac{24-15}{5}$

$= \frac{9}{5}$

From (1)

$\mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+5s+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{8}{15}\frac{1}{s-1} + \frac{\left(-\frac{8}{15}\right)s + \frac{9}{5}}{s^2+5s+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{8}{15}\frac{1}{s-1} + \frac{\left(-\frac{8}{15}\right)s + \frac{9}{5}}{s^2+2\cdot s\cdot\frac{5}{2}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{8}{15}\frac{1}{s-1} + \frac{\left(-\frac{8}{15}\right)s + \frac{9}{5}}{\left(s+\frac{5}{2}\right)^2+\frac{11}{4}}\right\}$

Long Exercise Book

$$= \frac{8}{15} L^{-1}\left(\frac{1}{s-1}\right) + \left(\frac{-8}{15}\right) L^{-1}\left\{\frac{s}{\left(s+\frac{5}{2}\right)^2 + \frac{\sqrt{11}}{2}}\right\}$$

$$+ \frac{9}{5} \quad L^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^2 + \frac{\sqrt{11}}{2}}\right\}$$

$$= \frac{8}{15} e^{t} - \frac{8}{15} e^{\frac{-5}{2}t} \quad L^{-1}\left\{\frac{s}{\left(s^2 + \frac{\sqrt{11}}{2}\right)}\right\}$$

$$+ \frac{9}{5} e^{-\frac{5}{2}t} L^{-1}\left\{\frac{1}{s^2 + \frac{\sqrt{11}}{2}}\right\}$$

$$= \frac{8}{15} e^{t} - \frac{8}{15} e^{-\frac{5}{2}t} \cos\frac{\sqrt{11}}{2}t + \frac{9}{5} e^{-\frac{5}{2}t} \sin\frac{\sqrt{11}}{2}t$$

(Ans)

Practice problems

Find laplace Inverse of following functions

1) $\frac{1}{s^2(s+2)}$ (2) $\frac{1}{(s+2)^2(s-2)}$

(3) $\frac{s+3}{(s^2+6s+13)^2}$ (4) $\tan^{-1}\left(\frac{1}{s}\right)$

(5) $\log\left(\frac{1+s}{s}\right)$ (6) $\frac{s+2}{(s^2+4s+5)^2}$

(7) $\log\frac{s+1}{(s+2)s+3}$ (8) $\frac{3(s^2-2)^2}{2s^5}$

(8) $\frac{s}{s^2+6s+13}$ (9) $\frac{1}{(s^2+a^2)^2}$

⟨10⟩ $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ (11) $\frac{1}{s(s-a)}$

⟨12⟩ $\frac{1}{s(s+2)^3}$ (13) $\frac{5s-2}{s^2(s+2)(s-1)}$

(14) $\frac{s^2+s+3}{(s+1)^2(s-3)}$ (15) $\frac{3s+7}{s^2-2s-3}$

16⟩ $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ ⟨17⟩ $\frac{3}{s^3-8}$

⟨18⟩ $\frac{3s-4}{16-s^2}$ ~~⟨19⟩ $\frac{s+2}{s^2+4s+5}$~~

⟨19⟩ $\tan^{-1}\left(\frac{1}{s}\right)$

⟨20⟩ $s\log\left(\frac{s+4}{s-4}\right)$